Math 275D Lecture 21 Notes

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1 Itô Integration of L^2_{loc} Functions and Local Martingales

1.1 Why only L^2_{loc}?

Why should we not try to integrate functions in L^p_{loc} for $p \neq 2$?

Proposition 1.1. *Let* $f(t) := \int_0^t (1-s)^{-1/2} \, dB_s$. *Then* $\lim_{s \to 1} f(s)$ *does not exist.*

Proof. The idea is that $f(t_2) - f(t_1) \perp f(t_1) - f(t_3)$ if $(t_1, t_2) \cap (t_3, t_4) = \varnothing$. If we look at $\|f(t)\|_{L^2}$ for fixed t, this goes to ∞ as $t \to 1$.

$$\|f(t)\|_{L^2} = \int_0^t (1-s)^{-1} \, ds \xrightarrow{t \to 1} \infty.$$

Then let t_k be such that $\|f(t_k) - f(t_{k-1})\|_{L^2} \geq 2\|f(t_{k-1})\|_{L^2}$. $\square$

1.2 Defining the Itô integral for L^2_{loc} functions

If $f \in L^2_{\text{loc}}$, we want to define the Itô integral

$$F_t = \int_0^t f \, dB_s.$$

We know that

$$\mathbb{P}\left(\int_0^T f^2(t, \omega) \, dt < \infty \right) = 1.$$

The idea is to define $f^{(n)}(t) = f(t \wedge \tau_n)$, where $\tau_n := \inf\{r : \int_0^r f^2 \, ds \geq n\}$. Then

$$\mathbb{E}\left[\int_0^T (f^{(n)})^2 \, dt \right] < \infty.$$

Now, we can let

$$F_t^{(n)} = \int_0^t f^{(n)} \, dB_s, \qquad F_t = \lim_{n \to \infty} F_t^{(n)}.$$

We get that $f(t)^{(n+1)} = f(t)^{(n)}$ for $t \leq \tau_n$. From our considerations last time, this gives $F^{(n+1)}(t) = F^{(n)}(T)$ for $t \leq \tau_n$. And since $f \in L^2_{\text{loc}}$, $T \wedge \tau_n \to T$.

Moreover, F_t is a continuous function (which we get from the sequential consistency $F^{(n+1)}(t) = F^{(n)}(T)$ for $t \leq \tau_n$). In general we have the following, for any stopping times τ_n.

Proposition 1.2. *Let $f \in L^2_{\text{loc}}$, and let $\tau = (\tau_n)_n$ be stopping times with $\tau_n < \tau_m$ for $m > n$ and $T \wedge \tau_n \to T$. Then $f_n(t) := f_{t \wedge \tau_n}$ are $\mathcal{H}^2$ functions. With these stopping times,*

$$F_t^{(\tau)} := \lim_n F_t^{(n)},$$

where the convergence is uniform convergence on compact sets.

We want to show that this definition is independent of τ.

Proposition 1.3. *If τ and μ are families of stopping times satisfying these properties, then $F^{(\tau)} = F^{(\nu)}$. a.s.*

Proof. Define $\mathcal{W}_n = \min\{\tau_n, \nu_n\}$. Then $F_t^{(n),\tau} = F_t^{(n),\nu}$ for all $t \leq \mathcal{W}_n$. $\square$

1.3 Local martingales

If $f \in \mathcal{H}$, then $\int f\, dB_s$ is a martingale. How about for $f \in L^2_{\text{loc}}$? It is not, but it is a local martingale.

Definition 1.1. Let X_t be a continuous random process with stopping times $\tau_n \nearrow +\infty$ and let $X_t^{(n)} = X_{t \wedge \tau_n}$. (X_t, τ) is a **local martingale** if $X_t^{(n)}$ is a martingale for each n.

Here is a natural example of a local martingale which is not a martingale.

Example 1.1. A simple random walk is a martingale, and we can linearly interpolate to get a continuous random process X_t. Look at $\tau := \inf\{n : S_n = -1\}$. Then $\tau < \infty$ a.s., but

$$\mathbb{E}[\lim_{t \to \infty} X_{t \wedge \tau}] = -1 \neq \lim_{t \to \infty} \mathbb{E}[X_{t \wedge \tau}].$$

If $f(t)$ is increasing and continuous and X_t is a martingale, then $Z_t := X(f(t))$ is still a martingale. We are going to use this change the scale of the times. Let $Y_t = B_{t \wedge \tau}$, where $\tau = \inf\{t : B_t \leq -1\}$. Now define

$$X_t = Y_{\frac{t}{1-t}}, \qquad 0 < t < 1.$$

Then X_t is a martingale on $[0, 1)$. Now extend

$$X_t^+ := \begin{cases} X_t & 0 \leq t < 1 \\ -1 & t \geq 1. \end{cases}$$

We claim that X_t^+ is a local martingale. Define $\tau_n = \min(\inf\{t : X_t \geq n\}, n)$. Then $\tau_n \nearrow \infty$. Then $X^+_{t \wedge \tau_n}$ is a bounded martingale.